Quantum Mechanics with Complex Time : A Comment to the Paper by Rajeev

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Abstract

In (quant–ph/0701141) Rajeev studied quantization of the damped simple harmonic oscillator and introduced a complex–valued Hamiltonian (which is normal). In this note we point out that the quantization is interpreted as a quantum mechanics with **complex time**. We also present a problem on quantization of classical control systems.

In the paper [1] Rajeev studied a quantization of the damped simple harmonic oscillator and introduced a complex–valued Hamiltonian (which is normal). In the case where dissipative systems are treated complex–valued Hamiltonians cannot be avoided in general.

In this note we point out that the quantization with complex–valued Hamiltonian is interpreted as a quantum mechanics with complex time, which is in a certain sense reasonable.

Anyway, quantization of dissipative systems must be widely studied.

Quantum Mechanics with Complex Hamiltonian

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First of all let us review the result in [1] within our necessity. In [1] there are some typos, so we give an explicit expression as much as we can.

The (differential) equation of the damped simple harmonic oscillator is given by

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0, \quad \gamma > 0 \tag{1}$$

where $x = x(t)$, $\dot{x} = dx/dt$ and the mass is set to 1 for simplicity. In the following we treat only the case $\omega > \gamma$ (the case $\omega = \gamma$ may be interesting).

The general solution is well–known to be

$$x(t) = A\mathrm{e}^{-\gamma t}\sin(\omega_1 t + \theta) \tag{2}$$

where A and θ are real constants (see (5) as to ω_1).

A comment is in order. If $\omega = \gamma$ or $\omega_1 = 0$ then the solution is

$$x(t) = \mathrm{e}^{-\gamma t}(c_0 + c_1 t)$$

where c_0 and c_1 are real constants. However, in the paper we don't treat this case.

Let us rewrite the equation (1)

$$\dot{x} = p, \quad \dot{p} = -\omega^2 x - 2\gamma p \tag{3}$$

or in the matrix form

$$\frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\gamma \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} \iff \frac{d}{dt}X = AX. \tag{4}$$

Next, let us make A diagonal. The characteristic equation of A is

$$0 = |\lambda E - A| = \begin{vmatrix} \lambda & -1 \\ \omega^2 & \lambda + 2\gamma \end{vmatrix} = (\lambda + \gamma)^2 + \omega^2 - \gamma^2 \equiv (\lambda + \gamma)^2 + {\omega_1}^2$$

, from which we have

$$\lambda_- = -\gamma - i\omega_1, \quad \lambda_+ = -\gamma + i\omega_1; \qquad \omega_1 = \sqrt{\omega^2 - \gamma^2}. \tag{5}$$

By the usual procedure, if we set

$$U = \frac{1}{\sqrt{2\omega_1}}(|\lambda_-),|\lambda_-)) = \frac{1}{\sqrt{2\omega_1}}\begin{pmatrix} 1 & 1 \\ \lambda_- & \lambda_+ \end{pmatrix} \Longrightarrow U^{-1} = \frac{-i}{\sqrt{2\omega_1}}\begin{pmatrix} \lambda_+ & -1 \\ -\lambda_- & 1 \end{pmatrix}$$

then

$$A = U\begin{pmatrix} \lambda_- & \\ & \lambda_+ \end{pmatrix}U^{-1} \equiv UD_AU^{-1}$$

and from (4)

$$\frac{d}{dt}X = AX \Longrightarrow \frac{d}{dt}U^{-1}X = D_AU^{-1}X.$$

Therefore new variables are

$$\begin{pmatrix} z \\ z^* \end{pmatrix} \equiv U^{-1}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2\omega_1}}(\omega_1 x + i(p+\gamma x)) \\ \frac{1}{\sqrt{2\omega_1}}(\omega_1 x - i(p+\gamma x)) \end{pmatrix} \tag{6}$$

and

$$\frac{dz}{dt} = \lambda_- z, \quad \frac{dz^*}{dt} = \lambda_+ z^*. \tag{7}$$

The equations have some deep meaning as shown in the latter.

The Poisson bracket is defined as

$$\{F,G\} = \frac{\partial F}{\partial x}\frac{\partial G}{\partial p} - \frac{\partial G}{\partial x}\frac{\partial F}{\partial p} \tag{8}$$

for $F = F(x,p)$ and $G = G(x,p)$. This gives $\{x,p\} = 1$, and therefore $\{z^*,z\} = i\{x,p\} = i$ from (6). Here, if we define a complex–valued Hamiltonian

$$\mathcal{H} = (\omega_1 - i\gamma)zz^* \tag{9}$$

then the equations (7) are easily recovered

$$\frac{dz}{dt} = \lambda_- z = -i(\omega_1 - i\gamma)z = \{z,\mathcal{H}\}, \quad \frac{dz^*}{dt} = \lambda_+ z^* = i(\omega_1 + i\gamma)z^* = \{z^*,\mathcal{H}^*\}.$$

In this stage we cannot avoid a complex Hamiltonian.

A comment is in order. The limit $\gamma \to 0$ recovers the usual Hamiltonian $\mathcal{H} = \omega zz^*$, while the limit $\gamma \to \omega$ gives $\mathcal{H} = -i\omega zz^*$ (pure imaginary).

Next, let us turn to the canonical quantization of the system. The usual procedure is given by the correspondence (representation) $z \to a^\dagger$ and $z^* \to \hbar a$ with $[a, a^\dagger] = 1$. Then the (quantized) Hamiltonian becomes

$$\mathcal{H} = (\omega_1 - i\gamma)zz^* = (\omega_1 - i\gamma)\frac{zz^* + z^*z}{2} \longrightarrow H = \hbar(\omega_1 - i\gamma)(a^\dagger a + 1/2), \tag{10}$$

from which the energy spectrum appears to be

$$\hbar(\omega_1 - i\gamma)(n + 1/2) = \hbar(\omega_1 - i\gamma)n + \frac{\hbar(\omega_1 - i\gamma)}{2}$$

for $n \geq 0$. However, this is not true. We must impose a restriction on the ground state energy. Namely, it is free of γ, which is reasonable from the physical point of view. Therefore

$$\left.\frac{\hbar(\omega_1 - i\gamma)}{2}\right|_{\gamma=0} = \frac{\hbar\omega}{2},$$

so the real spectrum[1] is

$$\hbar(\omega_1 - i\gamma)n + \frac{\hbar\omega}{2} \quad \text{for} \quad n \geq 0. \tag{11}$$

Let us consider the time evolution of our system. For any initial state

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n\rangle \quad \text{where} \quad \sum_{n=0}^{\infty} |\psi_n|^2 = 1$$

the time evolution is given by

$$|\psi\rangle \longrightarrow |\psi(t)\rangle \equiv \mathrm{e}^{-itH}|\psi\rangle = \mathrm{e}^{-it\frac{\hbar\omega}{2}} \sum_{n=0}^{\infty} \psi_n \mathrm{e}^{-it\hbar(\omega_1 - i\gamma)n}|n\rangle \tag{12}$$

from (11). Since

$$\sum_{n=0}^{\infty} \psi_n \mathrm{e}^{-it\hbar(\omega_1 - i\gamma)n}|n\rangle = \sum_{n=0}^{\infty} \psi_n \mathrm{e}^{-t\hbar\gamma n}\mathrm{e}^{-it\hbar\omega_1 n}|n\rangle,$$

the limit $t \to \infty$ gives

$$|\psi(t)\rangle \longrightarrow \mathrm{e}^{-it\frac{\hbar\omega}{2}}\psi_0|0\rangle. \tag{13}$$

All states will settle down to the ground state after infinite time. Compared with the classical result (2) all this sounds reasonable as Rajeev says.

[1] This point is not clear in [1]

Quantum Mechanics with Complex Time

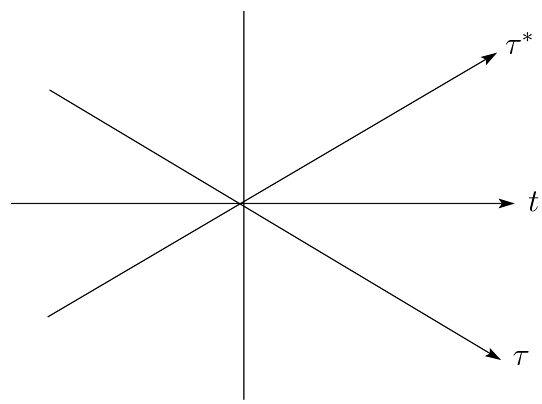
In this section we point out that it is possible to transform the preceding result into the usual form with complex time.

From (7) with (5) we have

$$\frac{dz}{dt} = (-\gamma - i\omega_1)z = \left(1 - i\frac{\gamma}{\omega_1}\right)(-i\omega_1)z, \quad \frac{dz^*}{dt} = (-\gamma + i\omega_1)z^* = \left(1 + i\frac{\gamma}{\omega_1}\right)(i\omega_1)z^*.$$

From this it is reasonable to introduce a complex time

$$\tau = \left(1 - i\frac{\gamma}{\omega_1}\right)t \implies \tau^* = \left(1 + i\frac{\gamma}{\omega_1}\right)t, \tag{14}$$



so the equations above reduce to

$$\frac{dz}{d\tau} = -i\omega_1 z, \quad \frac{dz^*}{d\tau^*} = i\omega_1 z^*. \tag{15}$$

If we define a (usual) Hamiltonian

$$\widetilde{\mathcal{H}} = \omega_1 z z^* \tag{16}$$

then we can recover

$$\frac{dz}{d\tau} = -i\omega_1 z = \{z, \widetilde{\mathcal{H}}\}, \quad \frac{dz^*}{d\tau^*} = i\omega_1 z^* = \{z^*, \widetilde{\mathcal{H}}\}.$$

Therefore the usual system of harmonic oscillator is obtained except for complex time.

The quantum Hamiltonian is

$$\widetilde{H} = \hbar\omega_1(a^\dagger a + 1/2) \tag{17}$$

and the "time" evolution is

$$|\psi\rangle \longrightarrow |\psi(\tau)\rangle \equiv \mathrm{e}^{-i\tau\widetilde{H}}|\psi\rangle = \mathrm{e}^{-it\frac{\hbar\omega}{2}}\sum_{n=0}^{\infty}\psi_n \mathrm{e}^{-i\tau\hbar\omega_1 n}|n\rangle, \tag{18}$$

$$|\psi\rangle \longrightarrow |\psi(\tau^*)\rangle \equiv \mathrm{e}^{i\tau^*\widetilde{H}}|\psi\rangle = \mathrm{e}^{it\frac{\hbar\omega}{2}}\sum_{n=0}^{\infty}\psi_n \mathrm{e}^{i\tau^*\hbar\omega_1 n}|n\rangle \tag{19}$$

for $|\psi\rangle = \sum_{n=0}^{\infty}\psi_n|n\rangle$, where we again assumed that the ground state energy is independent of γ (or independent of imaginary part, $\mathrm{Re}\ \tau = \mathrm{Re}\ \tau^* = t$).

Again, all this sounds reasonable.

Now, we present a problem. One of the simplest examples of classical control system is

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = f(t), \quad \gamma > 0 \tag{20}$$

where $f(t)$ is some outer field that we can control. In the matrix form

$$\frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\gamma \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} + f(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{21}$$

Since the system has been quantized if $f(t) = 0$, construct a theory to control the quantized system under $f(t)$. Note that $f(t)$ is a classical field and not a quantized one.

In last, let us conclude the paper by stating the motivation. We are studying a quantum computation based on Cavity QED, see for example [2] and [3]. To construct a more realistic model of quantum computer we must take **decoherence** into consideration, which is however a very hard problem. How to control decoherence has not been known as far as we know. Much work is required !

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References

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